

# Anomalous Monopole In an Interacting Boson System

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(Dated: February 19th, 2008)

Anomalous monopole of disk shape is found to exist in the semiclassical theory of a two-mode interacting boson system. The quantum origin of this anomaly is the collapsing or bundling of field lines of Berry curvature caused by the interaction between bosons in the semiclassical limit. The significance of this anomalous monopole is twofold: (1) it signals the failure of the von Neumann-Wigner theorem in the semiclassical limit; (2) it indicates a breakdown of the correspondence principle between quantum and classical dynamics.

PACS numbers: 03.65.Sq, 03.65.Vf, 05.45.Mt, 03.75.-b

Magnetic monopole was first discussed by Dirac as a quantization condition for electric charge[1]. Although it has never been observed in experiment as an fundamental particle, the monopole has fascinated physicists ever since[2]. Interestingly, monopoles have attracted great attention in a very different context as degeneracies or diabolical points of energy levels in parameter space[3, 4, 5]. The examples include the degeneracy of Bloch bands in the Brillouin zone[6] and energy levels in molecular magnets[7, 8]. The monopoles in this context are found to be crucial to understanding these systems.

In this work we study a two-mode interacting boson system that depends on three external parameters. For simplicity, we focus on its ground state, which is doubly degenerate at one isolated point in the parameter space. We find that the field lines of Berry curvature emanating from the point are curved due to the interaction between bosons. Moreover, at large  $N$  limit, that is, when the number of bosons  $N$  increases to infinity, the field lines collapse and bundle into a two-dimensional disk whose radius is determined by the interaction strength.

At large  $N$  limit, this boson system can be well described by a mean-field theory[9, 10]. Since this boson system belongs to a class of quantum systems which become classical at large  $N$  limit[11], this mean-field can be regarded as a semiclassical theory. We discover that the semiclassical (or mean-field) ground state of this boson system is degenerate at every point on the two-dimensional disk mentioned above. This means that the whole disk is a monopole. This is in stark contrast with what is demanded by the von Neumann-Wigner theorem[12]: the monopole in a three-dimensional parameter space is always a point-like object. Therefore, this anomalous monopole of disk shape indicates that the von Neumann-Wigner theorem fails in the semiclassical limit. Our further analysis shows that the magnetic charge is not uniformly distributed in the disk while its total charge is still  $2\pi$ , the Chern number[5]. In addition, this anomalous monopole is compared to an anomalous

monopole that is formed in a trivial fashion.

The Berry curvatures are computed for this system within the semiclassical theory and compared to the results in the quantum description. The matching becomes better as  $N$  increases as expected from the correspondence principle between quantum and classical dynamics[13]. However, on the monopole disk, the Berry curvature differs significantly between its semiclassical result and quantum result even in the large  $N$  limit. This shows that the existence of the anomalous monopole indicates a breakdown of the correspondence principle. This breakdown is analyzed from a fresh perspective by regarding the three external parameters as the dynamical variables of a massive classical particle.

The two-mode interacting boson system is described by the following second quantized Hamiltonian

$$\hat{H}_N = \frac{X}{2}(\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger) + \frac{iY}{2}(\hat{a} \hat{b}^\dagger - \hat{a}^\dagger \hat{b}) + \frac{Z}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}) - \frac{\lambda}{4V}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})^2, \quad (1)$$

where  $\hat{a}^\dagger$ ,  $\hat{a}$  and  $\hat{b}^\dagger$ ,  $\hat{b}$  are bosonic operators for two different quantum states, respectively,  $\lambda > 0$  is the interacting strength between bosons, and  $V$  is the volume of the system. The three parameters,  $X, Y$ , and  $Z$ , characterize the influence from environment or another system. This Hamiltonian has its root in modeling the Bose-Einstein condensates in a double-well potential[10]. It also belongs to a class of Hamiltonians studied in Refs.[7, 8] for single molecule magnet if we introduce  $\hat{J}_x = (\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a})/2$ ,  $\hat{J}_y = i(\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a})/2$ , and  $\hat{J}_z = (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})/2$ . The focus of studies in Refs.[7, 8] is on the pattern and topological property of the monopoles. In this work we examine the “magnetic” fields, i.e., Berry curvatures, generated by the monopoles and their behavior in the semiclassical limit  $N \rightarrow \infty$ . Note that large  $N$  limit is always taken by keeping  $N/V$  constant.

For simplicity, we concentrate on the ground state of this system. At point  $X = Y = Z = 0$ , we have  $\hat{H}_N =$

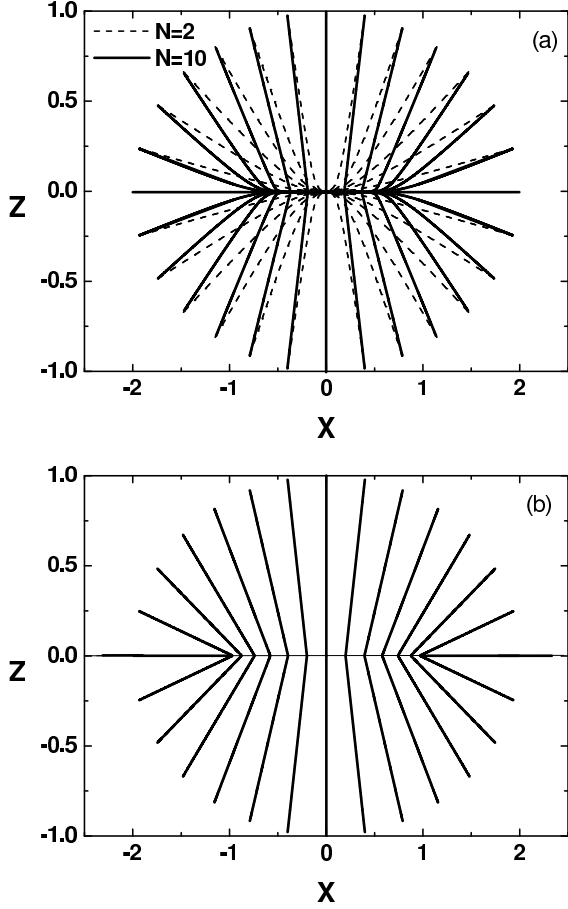


FIG. 1: Field lines of Berry curvature near the monopole for (a) the second-quantized Hamiltonian (dashed lines are for  $N = 2$  and solid lines are for  $N = 10$ ); (b) the mean-field Hamiltonian. Due to the symmetry around  $Z$ -axis, the  $Y$  component is omitted.  $c = 1$ . The lines in (b) are not straight as they appear.

$-\lambda(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})^2/4V$ , whose ground state is either  $\langle \hat{a}^\dagger \hat{a} \rangle = N$  or  $\langle \hat{b}^\dagger \hat{b} \rangle = N$ . This means that the ground state of this boson system is doubly degenerate at point  $X = Y = Z = 0$ . The ground state is not degenerate elsewhere in the parameter space.

This degenerate point or monopole at  $X = Y = Z = 0$  generates “magnetic” field  $\mathbf{B}_N$  (Berry curvature) in the parameter space spanned by  $X, Y, Z$ . As one usually uses field lines to illustrate a magnetic field, we have computed numerically the field lines of Berry curvature and plotted them in Fig.1(a). For clarity, only the results for  $N = 2$  and  $N = 10$  are plotted. Nevertheless an interesting trend is clearly demonstrated: the field lines are curved towards a disk defined by  $\sqrt{X^2 + Y^2} = c = N\lambda/V$  and  $Z = 0$ ; the curving gets stronger as  $N$  increases. In fact, our numerical results show that the field lines will collapse and bundle (or converge) into the disk when  $N$  approaches infinity. As we know, a magnetic monopole

(or an electric charge) can be viewed as the converging point or the emitting source of field lines. This collapsing (or converging) of field lines suggests that the whole disk become a monopole in the limit  $N \rightarrow \infty$ . This is indeed the case as we shall show.

At large  $N$  limit, this boson system becomes “classical” and can be described by the following mean-field (or semiclassical) Hamiltonian[9, 10],

$$H_s = \lim_{N \rightarrow \infty} \frac{\hat{H}_N}{N} = \frac{X}{2}(a^*b + ab^*) + \frac{iY}{2}(ab^* - a^*b) + \frac{Z}{2}(|a|^2 - |b|^2) - \frac{c}{4}(|a|^2 - |b|^2)^2, \quad (2)$$

where  $a$  and  $b$  are complex amplitudes for the system in the two quantum modes. The normalization is one, i.e.,  $|a|^2 + |b|^2 = 1$ . This kind of nonlinear Hamiltonian also appears in photoassociation systems[14, 15].

Within this semiclassical description, the ground state of this system is given by

$$|\phi\rangle \equiv \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1-p}{2}} \\ -\sqrt{\frac{1+p}{2}} \frac{X+iY}{\sqrt{X^2+Y^2}} \end{pmatrix}, \quad (3)$$

where  $p$  is the solution of the following equation,

$$p\sqrt{X^2 + Y^2} = (Z + cp)\sqrt{1 - p^2}. \quad (4)$$

This equation has one real root when  $\sqrt{X^2 + Y^2} \geq c$ . When  $X^2 + Y^2 < c^2$ , it can have three real roots. In particular, when  $Z = 0$ , two of the three real roots given by  $p = \pm \sqrt{1 - (X^2 + Y^2)/c^2}$  have the same energy and are for the ground states. This means that in the semiclassical description of the system, the ground state is degenerate on the disk given by  $X^2 + Y^2 < c^2$  and  $Z = 0$ . In other words, the whole disk is a monopole (see Fig.2).

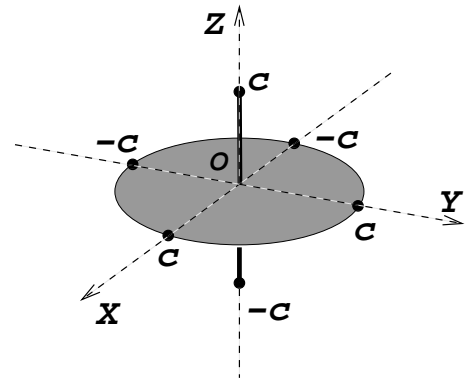


FIG. 2: Anomalous monopoles. The disk is the monopole for the ground state and the thick vertical line is the monopole for the highest eigenstate.

This anomalous disk-shaped monopole is very surprising. According to the von Neumann-Wigner theorem,

the accidental degeneracy of a quantum system occurs only at isolated points in a three dimensional parameter space[12]. In other words, the monopole must be a point. This is indeed the case when our system is described by the second-quantized Hamiltonian: the ground state is degenerate only at point  $X = Y = Z = 0$  as we have already pointed out. However, in the semiclassical description, that is, at large  $N$  limit, the monopole is a two-dimensional disk as the result of the curving and collapsing of field lines. This shows by example that the von Neumann-Wigner theorem does not hold in the semiclassical limit.

The significance of this anomalous monopole can be appreciated from a different angle. We consider the highest eigenstate of this system. For this eigenstate, its semiclassical monopole is a line as shown in Fig.2. However, this line-shaped monopole which appears also anomalous is trivial, not as significant as the disk-shaped monopole. The reason is as follows. In the second quantized model, the highest eigenstate has  $N$  equally spaced degenerate points along  $Z$  axis between  $-c$  and  $c$ . As  $N$  approaches infinity, these degenerate points merge into a line, forming in a trivial fashion the one dimensional monopole.

Let us now examine this disk-shaped anomalous monopole in detail. Although the semiclassical Hamiltonian  $H_s$  is nonlinear, the Berry curvature  $\mathcal{B}$  of this monopole can be computed as in a linear system[16]. That is to compute the curl of the vector potential  $\mathbf{A} = \langle \phi | \nabla | \phi \rangle$  with  $|\phi\rangle$  given in Eq.(3). The Berry curvature  $\mathcal{B}$  is found to be

$$\mathcal{B} = \frac{p^3}{2(cp + Z)^2(cp^3 + Z)}(\mathbf{R} + cp\hat{z}), \quad (5)$$

where  $\mathbf{R} = \{X, Y, Z\}$  and  $\hat{z}$  is the unit vector for  $Z$  direction. This result is plotted as field lines in Fig.1(b). It is apparent that these semiclassical field lines away from the monopole disk are very similar to the field lines obtained with the second quantized model. Note that  $\mathcal{B}$  has two different values on the monopole disk due to the double degeneracy of the ground state. By integrating  $\mathcal{B}$  over a closed surface around a small area in the disk, we find that the “magnetic” charge is not uniformly distributed over the disk. The charge distribution is

$$\rho = \frac{1}{c\sqrt{c^2 - (X^2 + Y^2)}}. \quad (6)$$

The integration of this charge density over the whole disk gives us a Chern number of  $2\pi$ . So, although the monopole has changed from a point to a disk as the semiclassical limit is approached, the total charge does not change. It is worthwhile to mention that the total charge of the line-shaped monopole is infinite as easily inferred from its trivial origin.

Berry[13] once established a semiclassical relation between Berry phase[4] and Hannay’s angle[17, 18] in ac-

cordance with the correspondence principle. This semiclassical relation basically says that the two-forms for Berry phase and Hannay’s angle (the two-form for Berry phase is the usual Berry curvature) are the same in the semiclassical limit  $\hbar \rightarrow 0$ . This semiclassical relation should hold in this interacting boson system. We define two pairs of conjugate variables,  $p_a = \sqrt{i\hbar}a^*$ ,  $q_a = \sqrt{i\hbar}a$  and  $p_b = \sqrt{i\hbar}b^*$ ,  $q_b = \sqrt{i\hbar}b$  for the semiclassical Hamiltonian[19]. The quantization is realized with the following commutators,

$$[\hat{q}_a, \hat{p}_a] = [\hat{q}_b, \hat{p}_b] = i\hbar/N. \quad (7)$$

One can obtain the second quantized Hamiltonian(1) with the following substitution  $\hat{a} = \sqrt{N/i\hbar}\hat{q}_a$ ,  $\hat{a}^\dagger = \sqrt{N/i\hbar}\hat{p}_a$  and  $\hat{b} = \sqrt{N/i\hbar}\hat{q}_b$ ,  $\hat{b}^\dagger = \sqrt{N/i\hbar}\hat{p}_b$ . These commutators show why  $N \rightarrow \infty$  is the semiclassical limit. As a result, the semiclassical relation established by Berry[13] for this boson system is

$$\lim_{N \rightarrow \infty} \delta \mathbf{B} = \lim_{N \rightarrow \infty} \left( \frac{\mathbf{B}_N}{N} - \mathcal{B} \right) = 0, \quad (8)$$

Note that the Hannay’s angle in the semiclassical system  $H_s$  is just the Berry phase generalized for nonlinear quantum system in Ref.[16] and the semiclassical Berry curvature  $\mathcal{B}$  is the two-form for this Hannay’s angle.

Our numerical results show that the relation (8) indeed holds almost everywhere in the parameter space except on the monopole disk. On the disk, the semiclassical Berry curvature  $\mathcal{B}$  has a non-zero  $\hat{z}$  component while the quantum  $\mathbf{B}_N$  always points radially in the  $Z = 0$  plane. Furthermore, the quantum Berry curvature  $\mathbf{B}_N$  diverges exponentially with  $N$  on the monopole disk while the in-plane component of the semiclassical  $\mathcal{B}$  does not. We define  $d = |\delta \mathbf{B}^l|$ , where the superscript  $l$  denotes the component of the vector parallel to the  $XY$  plane. The difference  $d$  is plotted in Fig.3, where we see the difference  $d$  increases exponentially with  $N$ . This diverging difference shows that the semiclassical relation in Eq.(8) is broken. Therefore, the disk-shaped monopole also signifies the breakdown of the corresponding principle between quantum mechanics and classical mechanics. In the following, we shall look into this breakdown from a very different angle, showing that some quantum effect remains in the semiclassical limit.

We treat the three parameters,  $\mathbf{R} = \{X, Y, Z\}$ , as dynamical variables of a massive and classical particle, whose Hamiltonian is  $H_c = \mathbf{P}^2/2M + U(\mathbf{R})$ . In this way we obtain a Born-Oppenheimer type system where a fast quantum system is coupled to a heavy and slow classical system[19]

$$H = \langle \Psi | \hat{H}_N | \Psi \rangle + H_c, \quad (9)$$

where  $|\Psi\rangle$  is a general wavevector for the boson system. It is reasonable to expect that at the large  $N$  limit we

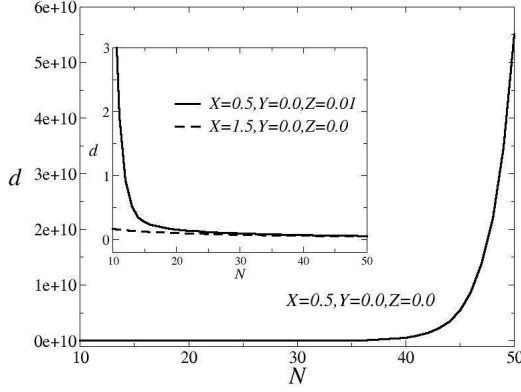


FIG. 3: The difference  $d = |\delta \mathbf{B}'|$  as a function of the number of bosons  $N$  at a point on the monopole disk. The inset shows the results for points away from the disk.  $c = 1$ .

can simplify the above Hamiltonian by replacing its first part by the semiclassical Hamiltonian in Eq.(2),

$$H' = NH_s + H_c. \quad (10)$$

However, in the following, we shall show that the two Hamiltonians  $H$  and  $H'$  do not have the same dynamics even in the limit  $N \rightarrow \infty$ . That is, there is a difference between the two Hamiltonians which does not vanish as  $N$  increases.

For a Born-Oppenheimer system, it is well known that the dynamics of the slow system  $H_c$  is controlled by two forces, Born-Oppenheimer force and geometric force[19, 20, 21]. The Born-Oppenheimer force  $\mathbf{F}_{BO} = -\nabla E_n(\mathbf{R})$  is the gradient of an eigenenergy of the fast system. If  $\mathbf{B}_g$  is the Berry curvature of the fast system, the dynamics of the slow system is then given by

$$M\ddot{\mathbf{R}} = \mathbf{F}_{BO} + \dot{\mathbf{R}} \times \mathbf{B}_g - \nabla U(\mathbf{R}). \quad (11)$$

One can use either the second quantized Hamiltonian  $\hat{H}_N$  or the semiclassical Hamiltonian  $H_s$  to compute both forces.

We consider a special case, where the slow classical particle is set with the initial condition,  $X = r < c$ ,  $Y = 0$ ,  $Z = 0$ , and  $\dot{X} = 0$ ,  $\dot{Y} = v$  and  $\dot{Z} = 0$  while the fast boson system is kept in its ground state. We set up ourselves to a task to design a potential  $U(\mathbf{R})$  so that the slow particle stays in  $Z = 0$  plane and makes a circular motion. When  $N$  is large, one would feel safe to design  $U(\mathbf{R})$  by using the semiclassical Hamiltonian  $H'$  to compute  $\mathbf{F}_{BO}$  and  $\mathbf{B}_g$ . However, due to the exponential breakdown of Eq.(8) discussed above, such designed  $U(\mathbf{R})$  will not be able to keep the classical particle in the  $Z = 0$  plane: the strong parallel component of  $\mathbf{B}_N$  will kick the particle out of the plane. This shows that there is always some physical consequence which can not be counted in the semiclassical theory. It is interesting note that Berry

and Robbins once pointed out that a kind of friction in a chaotic classical system does not exist in its corresponding quantum system and called it discordance[21]. What we observe here is similar to this discordance although our system is not chaotic in the semiclassical limit.

In conclusion, we have found an anomalous monopole of disk shape in a two-mode interacting boson system. This kind of anomalous monopole should exit in a general interacting boson system. For example, if one manages to compute the Bloch bands of an interacting boson system in a three dimensional periodic potential, one should expect such an anomalous monopole in the Brillouin zone. We have further demonstrated that this anomalous monopole is an indication of the failure of the von Neumann-Wigner theorem in the semiclassical limit and the breakdown of the correspondence principle between quantum and classical dynamics.

We thank the helpful discussion with Junren Shi and Qian Niu. This work is supported by the “BaiRen” program of Chinese Academy of Sciences, the NSF of China (10504040,10725521), and the 973 project of China(2005CB724500,2006CB921400).

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